The tuning problem
Statement: Given

- the **octave** $\omega$
- the **number of intervals** $n$
- the **intervals** $I_1, \ldots, I_n$
- the **interval weights** $i_1, \ldots, i_n$
- the **key weights** $\kappa_0, \ldots, \kappa_n$ (note one more than number of intervals)

The “matrix” $M$: We let $a_1, \ldots, a_n$ denote the keys that we are looking for, i.e., those frequency values that will make up an octave $0, a_1, \ldots, a_n, \omega$

give $M$, an $n$ by $n + 1$ matrix defined by

$$M = \begin{pmatrix}
a_1 - 0 & a_2 - a_1 & a_3 - a_2 & \ldots & a_n - a_{n-1} & \omega - a_n \\
a_2 - 0 & a_3 - a_1 & a_4 - a_2 & \ldots & \omega - a_{n-1} & \omega + a_1 - a_n \\
a_3 - 0 & a_4 - a_1 & a_5 - a_2 & \ldots & (\omega + a_1) - a_{n-1} & (\omega + a_2) - a_n \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
a_n - 0 & \omega - a_1 & (\omega + a_1) - a_2 & \ldots & (\omega + a_{n-2}) - a_{n-1} & (\omega + a_{n-1}) - a_n
\end{pmatrix}$$

Thus, thinking of the keys as "wrapped on the circle" (LP - there is a better way to say this, but I have to think about it) row 1 gives all the intervals from one key to the next, row 2 is all intervals of length 2, and so on.

The matrix $M$ simply organizes the intervals as indicated above. We are interested in an associated error function $E = E(a_1, \ldots, a_n)$ which measures the sum of squares of differences of each of the intervals from their ideal difference - these are the $I_j$:

$$E = [(m_{11} - I_1)^2 + \ldots + (m_{1,n+1} - I_1)^2 + \ldots + (m_{jk} - I_j)^2 + \ldots + (m_{n,n+1} - I_n)^2]$$

This is the unweighted version of the error function. The weighted version is

$$E = [i_1(\kappa_0 m_{11} - I_1)^2 + \ldots + i_1(\kappa_n m_{1,n+1} - I_1)^2 + \ldots + i_j(\kappa_{k-1} m_{jk} - I_j)^2 + \ldots + i_n(\kappa_n m_{n,n+1} - I_n)^2]$$
The problem of minimizing this can be done using the tools of multivariable calculus. This is a quadratic function of \( a_1, \ldots, a_n \). In one variable, a quadratic function is some kind of parabola. In order to find an extreme value, either a min or a max, the tools of calculus tell us - in one variable - to take a derivative, set the resulting function (the first derivative) equal to zero and then solve. At that value the original function takes on an extreme value (usually - there are other possibilities, but let’s forget that for now).

In several variables all the same ideas apply. However, in this case we need to take first partial derivatives with respect to each of the variables \( a_1, \ldots, a_n \), leading us to look at the set of simultaneous linear equations:

\[
\frac{\partial E}{\partial a_1} = 0 \\
\frac{\partial E}{\partial a_2} = 0 \\
\vdots \\
\frac{\partial E}{\partial a_n} = 0
\]

Which, assuming that things aren’t too bad, we solve. Note that it is \( n \) equations in \( n \) unknowns.

Specific example:

- the octave \( \omega = 1200 \)
- the number of intervals \( n = 2 \)
- the intervals \( I_1 = 400, I_2 = 800 \)
- the interval weights \( \iota_1 = 1, \iota_2 = 1 \)
- the key weights \( \kappa_0 = 1, \kappa_1 = 1, \kappa_2 = .2 \) (note one more than number of intervals)

\[
M = \begin{pmatrix}
a_1 & a_2 - a_1 & 1200 - a_2 \\
a_2 & 1200 - a_1 & (1200 + a_1) - a_2
\end{pmatrix}
\]

So that the unweighted error function is

\[
E_{unw} = [400 - a_1]^2 + [400 - (a_2 - a_1)]^2 + [400 - (1200 - a_2)]^2 + \\
[800 - a_2]^2 + [800 - (1200 - a_1)]^2 + [800 - ((1200 + a_1) - a_2)]^2
\]

\[
= [400 - a_1]^2 + [400 - a_2 + a_1]^2 + [a_2 - 800]^2 + \\
[800 - a_2]^2 + [a_1 - 400]^2 + [a_2 - a_1 - 400]^2
\]
and the weighted version is
\[ E_w = [400 - a_1]^2 + [400 - (a_2 - a_1)]^2 + [400 - .2(1200 - a_2)]^2 + [800 - a_2]^2 + [800 - (1200 - a_1)]^2 + [800 - .2((1200 + a_1) - a_2)]^2 \]
\[ = [400 - a_1]^2 + [400 - a_2 + a_1]^2 + [400 - .2(1200 - a_2)]^2 + [800 - a_2]^2 + [a_1 - 400]^2 + [800 - .2((1200 + a_1) - a_2)]^2 \]

Now we take partials with respect to \( a_1 \):
\[
\frac{\partial E_w}{\partial a_1} = -2[400 - a_1] + 2[400 - a_2 + a_1] + 2[a_1 - 400] + 2[800 - .2((1200 + a_1) - a_2)] * .2 \\
= -2[400 - a_1] + 2[400 - a_2 + a_1] + 2[a_1 - 400] + .4[800 - .2((1200 + a_1) - a_2)]
\]
and then \( a_2 \):
\[
\frac{\partial E_w}{\partial a_2} = -2[400 - a_2 + a_1] + 2[400 - .2(1200 - a_2)] * (-.2) \\
-2[800 - a_2] + 2[800 - .2((1200 + a_1) - a_2)] * .2 \\
= -2[400 - a_2 + a_1] - .4[400 - .2(1200 - a_2)] - 2[800 - a_2] + .4[800 - .2((1200 + a_1) - a_2)]
\]

and if we set these equal to zero we get a solution of
\[ a_1 = 412.698, \quad a_2 = 714.042 \]